

**UNCLASSIFIED**

---

**AD 296 795**

*Reproduced  
by the*

**ARMED SERVICES TECHNICAL INFORMATION AGENCY  
ARLINGTON HALL STATION  
ARLINGTON 12, VIRGINIA**



---

**UNCLASSIFIED**

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

63-2-4

NOLTR 62-177

CATALOGED BY ASTIA  
AS AD NO.

296795

NOL

A GENERALIZED BOUNDARY VALUE PROBLEM  
FOR  $u_{xy} = f(x, y, u, u_x, u_y)$

17 OCTOBER 1962



UNITED STATES NAVAL ORDNANCE LABORATORY, WHITE OAK, MARYLAND

NOLTR 62-177

296 795

- RELEASED TO ASTIA  
BY THE NAVAL ORDNANCE LABORATORY
- ☒ Without restrictions
  - ☐ For Release to Military and Government Agencies Only.
  - ☐ Approval by BuWeps required for release to contractors.
  - ☐ Approval by BuWeps required for all subsequent release.

Mathematics Department Report

A GENERALIZED BOUNDARY VALUE PROBLEM  
FOR  $u_{xy} = f(x, y, u, u_x, u_y)$

Prepared by:  
James Conlan

ABSTRACT: This paper proves an existence theorem for the partial differential equation  $u_{xy} = f(x, y, u, u_x, u_y)$  with certain linear combinations of  $u$ ,  $u_x$  and  $u_y$  given on two non characteristic curves.

PUBLISHED NOVEMBER 1962

U. S. NAVAL ORDNANCE LABORATORY  
WHITE OAK, MARYLAND

NOLTR 62-177


17 October 1962

A Generalized Boundary Value Problem for  $u_{xy} = f(x, y, u, u_x, u_y)$

This report deals with a generalization of the Coursat problem for  $u_{xy} = f(x, y, u, u_x, u_y)$ .

This work was performed under NOL Task No. FR-30.

ROBERT ODENING  
Captain, USN  
Commander



RICHARD C. ROBERTS  
By direction

CONTENTS

|   | Page |
|---|------|
| 1. INTRODUCTION . . . . .                             | 1    |
| 2. The boundary value problem . . . . .               | 3    |
| 3. The finite difference scheme . . . . .             | 5    |
| 4. The sequences of approximating functions . . . . . | 9    |
| 5. Convergence to a solution . . . . .                | 13   |
| 6. Other conditions on $f$ . . . . .                  | 15   |
| REFERENCES . . . . .                                  | 18   |

A GENERALIZED BOUNDARY VALUE PROBLEM  
FOR  $u_{xy} = f(x, y, u, u_x, u_y)$

INTRODUCTION

1. The Euler-Cauchy polygon method for proving the existence of a solution for

$$\frac{dy}{dx} = f(x, y), \quad y(0) = y_0,$$

was extended by J. B. Diaz [3] to the characteristic boundary value problem for a hyperbolic partial differential equation

$$(1.1) \quad \begin{aligned} u_{xy} &= f(x, y, u, u_x, u_y), \\ u(x, 0) &= \sigma(x), \quad u(0, y) = \tau(y), \quad \sigma(0) = \tau(0). \end{aligned}$$

Here  $f$  is assumed to be bounded and continuous for  $(x, y)$  in some rectangle  $R$ , and for all possible values of  $u, u_x, u_y$ , and to satisfy a Lipschitz condition in  $u_x, u_y$ . Under these conditions an existence theorem in the large, i.e., throughout all of  $R$ , was obtained.

This method has been further extended by I. I. Glick [4] to the  $n$ -dimensional analogue of (1.1),

$$u_{x_1 \dots x_n} = f, \quad u = \sigma_1 \quad \text{on } x_1 = 0,$$

where  $f$  is a function of the  $x_i$ 's of  $u$  and of all mixed partial derivatives of  $u$  of order less than  $n$ , and where the  $\sigma_i$ 's satisfy certain obvious compatibility conditions.

In the 2-dimensional case the method was extended by J. Conlan [2] to the Cauchy problem,

$$\begin{aligned}
 (1.2) \quad & u_{xy} = f(x, y, u, u_x, u_y) , \\
 & u_x = \sigma(x) \quad \text{on } y = x, \\
 & u_y = \tau(y) \quad \text{on } y = x, \\
 & u(0, 0) = 0 ,
 \end{aligned}$$

and to the mixed boundary value problem, i.e., boundary conditions in (1.2) replaced by

$$u(0, y) = \tau(y) , \quad u(x, x) = \sigma(x) , \quad \sigma(0) = \tau(0) .$$

In this paper we consider the 2-dimensional problem

$$\begin{aligned}
 (1.3) \quad & u_{xy} = f(x, y, u, u_x, u_y) , \\
 & u_x = \sigma + au_y + bu \quad \text{on } y = \eta(x) , \\
 & u_y = \tau + cu_x + du \quad \text{on } x = \xi(y) , \\
 & \eta(0) = \xi(0) = 0 , \\
 & u(0, 0) = u_0, \text{ a constant ,}
 \end{aligned}$$

where  $\sigma$ ,  $a$ ,  $b$  are continuous functions of  $x$ , and  $\tau$ ,  $c$ ,  $d$  are continuous functions of  $y$ . The method of proof used here is again by an extension of the method used in [3]. Using a different method, A. K. Aziz and J.B.Diaz[1]



have treated the corresponding problem for the linear equation  $u_{xy} + au_x + bu_y + cu = d$  where  $a, b, c$ , and  $d$  are functions only of  $x$  and  $y$ . We refer the reader to [1] for a discussion of the history of the problem, and for an extensive bibliography. In the linear case Aziz and Diaz were able to obtain an existence theorem in the large. However for problem (1.3) all we get is an existence theorem in the small.

We note that the characteristic boundary value problem, the Cauchy problem, and the mixed boundary value problem are special cases of (1.3). So also is the Goursat problem, i.e., boundary conditions in (1.3) replaced by

$$\begin{aligned} (1.4) \quad & u(x, \eta(x)) = \sigma(x) \quad , \\ & u(\xi(y), y) = \tau(y) \quad , \\ & u(0, 0) = \sigma(0) = \tau(0) = 0 \quad . \end{aligned}$$

To see this differentiate the first two of (1.4), obtaining

$$\begin{aligned} & u_x(x, \eta(x)) + \eta'(x)u_y(x, \eta(x)) = \sigma'(x) \quad , \\ & \xi'(y)u_x(\xi(y), y) + u_y(\xi(y), y) = \tau'(y) \quad , \end{aligned}$$

and these are of the same form as the boundary conditions of (1.3).

## 2. The boundary value problem.

Let  $R$  consist of the points  $(x, y)$  such that  $0 \leq x \leq A$  and  $0 \leq y \leq B$ . Let  $S$  consist of those points  $(u, p, q)$  such that  $-\infty < u, p, q < \infty$ . And let  $C$  be the cartesian product  $R \times S$ . The precise formulation of the problem to be solved is given in the following theorem:

Theorem 2.1. Hypotheses:

- (a)  $f$  is a continuous real valued function of  $(x, y, u, p, q)$  in  $C$ .
- (b) There is a constant  $M > 0$  such that  $|f(x, y, u, p, q)| < M$  in  $C$ .
- (c)  $f$  satisfies a uniform Lipschitz condition in  $(p, q)$ , i.e., there is a constant  $L > 0$  such that
 
$$|f(x, y, u, \bar{p}, \bar{q}) - f(x, y, u, p, q)| \leq L(|\bar{p} - p| + |\bar{q} - q|)$$
 whenever the arguments of  $f$  are in  $C$ .
- (d)  $\sigma, a, b, \eta$  are continuous functions of  $x$  for  $0 \leq x \leq A$ , and  $\tau, c, d, \xi$  are continuous functions of  $y$  for  $0 \leq y \leq B$ .
- (e)  $|a(x) \cdot c(y)| < 1$  for  $(x, y)$  in  $R$ .
- (f)  $\xi(0) = \eta(0) = 0$ , and  $0 \leq \xi(y) \leq A$ ,  $0 \leq \eta(x) \leq B$ .
- (g) There is a continuous function of  $x$ , say  $\zeta$ , such that  $\zeta(0) = 0$ ,  $\zeta(A) = B$ , and such that  $y = \zeta(x)$  has an inverse for  $0 \leq x \leq A$ . Moreover the graph of  $y = \eta(x)$  lies below that of  $y = \zeta(x)$  and the graph of  $x = \xi(y)$  lies above it, i.e.,  $\eta(x) \leq \zeta(x)$  and  $\xi(y) \leq \zeta^{-1}(y)$ .
- (h) Letting  $\|a\| = \sup \{|a(x)| : 0 \leq x \leq A\}$ , etc., we assume that
 
$$A \left\{ \frac{\|a\|}{1 - \|a\|} + \frac{\|d\|}{\|c\|} \right\} + B \left\{ \frac{\|b\|}{1 - \|a\|} + \frac{\|d\|}{\|c\|} \right\} < 1.$$
- (i)  $\|a\| \|c\| \exp L(A+B) < 1$ .

Conclusion:

There is at least one real valued function  $u(x, y)$  which is continuous together with its partial derivatives  $u_x, u_y, u_{xy}$  in  $R$ , and which satisfies the partial differential equation

$$u_{xy}(x, y) = f[x, y, u(x, y), u_x(x, y), u_y(x, y)]$$

and which satisfies the following boundary conditions (here we use the notation  $u_1(x,y) = u_x(x,y)$  and  $u_2(x,y) = u_y(x,y)$ ),

$$u_1[x, \eta(x)] = \sigma(x) + a(x) u_2[x, \eta(x)] + b(x) u[x, \eta(x)] ,$$

$$u_2[\xi(y), y] = \tau(y) + c(y) u_1[\xi(y), y] + d(y) u[\xi(y), y] ,$$

$$u(0,0) = u_0 .$$

The proof of this theorem will occupy the next several sections.

We note that the above boundary value problem is equivalent to the following system of integral equations, where  $F(x,y) = f[x,y,u(x,y), u_1(x,y), u_2(x,y)]$ .

$$\begin{aligned} (2.1) \quad u(x,y) &= \int_0^x \left\{ \sigma(s) + a(s) u_2[s, \eta(s)] + b(s) u[s, \eta(s)] \right\} ds \\ &+ \int_0^y \left\{ \tau(t) + c(t) u_1[\xi(t), t] + d(t) u[\xi(t), t] \right\} dt \\ &+ \left\{ \int_0^y dt \int_{\xi(t)}^x ds - \int_0^x ds \int_0^{\eta(s)} dt \right\} F(s,t) + u_0 , \\ u_1(x,y) &= \sigma(x) + a(x) u_2[x, \eta(x)] + b(x) u[x, \eta(x)] + \int_{\eta(x)}^y F(x,t) dt , \\ u_2(x,y) &= \tau(y) + c(y) u_1[\xi(y), y] + d(y) u[\xi(y), y] + \int_{\xi(y)}^x F(s,y) ds . \end{aligned}$$

### 3. The finite difference scheme.

Let  $n$  be a positive integer and let  $0 = x_0 < x_1 < \dots < x_n = A$ . Let  $y_j = \zeta(x_j)$  for  $j = 0, \dots, n$ , where  $\zeta$  is given in condition (g) of theorem 2.1. Then for  $0 \leq j, k < n-1$  let

$$R_{j,k} = \{ (x,y) : x_j \leq x < x_{j+1}, y_k \leq y < y_{k+1} \} ,$$

and

$$R_{n-1,k} = \{ (x,y): x_{n-1} \leq x \leq x_n, y_k \leq y < y_{k+1} \},$$

etc.

We next introduce the following functions. For  $(x,y)$  in  $R_{j,k}$  let

$$\lambda(x) = \begin{cases} x_{j-1} & \text{if } j > 0 \\ x_j & \text{if } j = 0 \end{cases}, \quad \mu(y) = \begin{cases} y_{k-1} & \text{if } k > 0 \\ y_k & \text{if } k = 0 \end{cases},$$

$$\hat{\pi}(x,y) = \sigma(x) + a(x)\rho(x,y) + b(x) \Theta(x,y),$$

$$\hat{\rho}(x,y) = \tau(y) + c(y)\pi(x,y) + d(y) \Theta(x,y),$$

$$\Theta(x,y) = \int_0^x \hat{\pi}[\lambda(s), \eta(\lambda(s))] ds + \int_0^y \hat{\rho}[\xi(\mu(t)), \mu(t)] dt$$

$$+ \left\{ \int_0^y dt \int_{\xi[\mu(t)]}^x ds - \int_0^x ds \int_0^{\eta[\lambda(s)]} dt \right\} \phi(s,t) + u_0,$$

$$\pi(x,y) = \begin{cases} (\Theta_{j+1,k} - \Theta_{j,k}) / (x_{j+1} - x_j), & j \geq 0, k \geq 1, \\ (\Theta_{j,0} - \Theta_{j-1,0}) / (x_j - x_{j-1}), & j \geq 1, k = 0, \end{cases}$$

where  $\Theta_{j,k} = \Theta(x_j, y_k)$ , etc.,

$$\rho(x,y) = \begin{cases} (\Theta_{j,k+1} - \Theta_{j,k}) / (y_{k+1} - y_k), & j \geq 1, k \geq 0, \\ (\Theta_{0,k} - \Theta_{0,k-1}) / (y_k - y_{k-1}), & j = 0, k \geq 1. \end{cases}$$

For  $(x,y)$  in  $R_{0,0}$  let  $\pi(x,y) = u_1(0,0)$ ,  $\rho(x,y) = u_2(0,0)$  where

$$u_1(0,0) = \sigma(0) + a(0) u_2(0,0),$$

$$u_2(0,0) = \tau(0) + c(0) u_1(0,0).$$

Let

$$\phi(x,y) = f[\lambda(x), \mu(y), \Theta(\lambda(x), \mu(y)), \pi(x,y), \rho(x,y)].$$

One can now verify that if  $\Theta$ ,  $\pi$ ,  $\rho$  have been computed in cells  $R_{i,j}$  for  $0 \leq i, j \leq k-1$ , then it is possible to compute  $\Theta$ ,  $\pi$ ,  $\rho$  in cells  $R_{k,j}$  and  $R_{i,k}$  for  $0 \leq i, j \leq k$  in terms of quantities already computed in the previous cells.

If  $j \geq 0, k \geq 1$  then for  $(x,y)$  in  $R_{j,k}$

$$\begin{aligned}
 (3.1) \quad \pi(x,y) &= \pi_{j,k} = (\Theta_{j+1,k} - \Theta_{j,k}) / (x_{j+1} - x_j) \\
 &= \frac{1}{x_{j+1} - x_j} \left\{ \int_{x_j}^{x_{j+1}} \hat{\pi}[\lambda(s), \eta(\lambda(s))] ds \right. \\
 &\quad \left. + \left[ \int_0^{y_k} dt \int_{x_j}^{x_{j+1}} ds - \int_{x_j}^{x_{j+1}} ds \int_0^{\eta[\lambda(s)]} dt \right] \phi(s,t) \right\} \\
 &= \hat{\pi}[x_{j-1}, \eta(x_{j-1})] + \int_{\eta(x_{j-1})}^{y_k} \phi(x_j, t) dt,
 \end{aligned}$$

where  $x_{j-1}$  is to be replaced by  $x_j$  when  $j = 0$ . But

$$\begin{aligned}
 \frac{\partial}{\partial x} \Theta(x,y) &= \hat{\pi}[\lambda(x), \eta(\lambda(x))] + \int_{\eta[\lambda(x)]}^y \phi(x,t) dt \\
 &= \hat{\pi}[x_{j-1}, \eta(x_{j-1})] + \int_{\eta(x_{j-1})}^y \phi(x,t) dt.
 \end{aligned}$$

Since  $\phi(x,y) = \phi(x_j, y)$ , we have

$$\frac{\partial}{\partial x} \Theta(x, y_k) = \pi(x, y_k) \quad \text{if } x_j < x < x_{j+1} ,$$

$$\frac{\partial^+}{\partial x} \Theta(x_j, y_k) = \pi(x_j, y_k) .$$

In the case of  $j \geq 1$ ,  $k = 0$  we get

$$\frac{\partial^+}{\partial x} \Theta(x_{j-1}, y_k) = \pi(x_j, y_k) .$$

Similarly if  $j \geq 1$ ,  $k \geq 0$  then

$$(3.2) \quad \rho(x, y) = \rho(x_j, y_k) = (\Theta_{j, k+1} - \Theta_{j, k}) / (y_{k+1} - y_k)$$

$$= \frac{1}{y_{k+1} - y_k} \left\{ \int_{y_k}^{y_{k+1}} \hat{p}[\xi(\mu(t)), \mu(t)] dt \right.$$

$$\left. + \int_{y_k}^{y_{k+1}} dt \int_{\xi[\mu(t)]}^{x_j} \phi(s, t) ds \right\}$$

$$= \hat{p}[\xi(y_{k-1}), y_{k-1}] + \int_{\xi(y_{k-1})}^{x_j} \phi(s, y_k) ds ,$$

where  $y_{k-1}$  is to be replaced by  $y_k$  when  $k = 0$ . Then as above

$$\frac{\partial}{\partial y} \Theta(x_j, y) = \rho(x_j, y) \quad \text{if } y_k < y < y_{k+1} ,$$

$$\frac{\partial^+}{\partial y} \Theta(x_j, y_k) = \rho(x_j, y_k) .$$

If  $j = 0, k \geq 1$  then

$$\frac{\partial^+}{\partial y} \Theta(x_j, y_{k-1}) = \rho(x_j, y_k) .$$

We now introduce the functions  $p, q$  defined as follows:

$$p(x, y) = \begin{cases} \frac{\partial^+}{\partial x} \Theta(x, y), & x = x_j \\ \frac{\partial}{\partial x} \Theta(x, y), & x_j < x < x_{j+1} \end{cases}$$

$$q(x, y) = \begin{cases} \frac{\partial^+}{\partial y} \Theta(x, y), & y = y_k \\ \frac{\partial}{\partial y} \Theta(x, y), & y_k < y < y_{k+1} . \end{cases}$$

In what follows we will be interested in a sequence of subdivisions of  $R$  such that as  $n$  tends to infinity

$$(3.3) \quad \alpha_n = \sup \{ |x_{j+1} - x_j| + |y_{k+1} - y_k| : 0 \leq j < n, 0 \leq k < n \}$$

tends to zero.

To each  $n$  will correspond a particular  $\Theta, \pi, \rho, p, q$  etc., and when we wish to indicate this explicitly we will write  $\Theta^{(n)}, \pi^{(n)}, \rho^{(n)}, p^{(n)}, q^{(n)}$  etc.

#### 4. The sequences of approximating functions.

We now want to show that the sequences  $\{ \Theta^{(n)} \}, \{ p^{(n)} \}, \{ q^{(n)} \}$  contain subsequences  $\{ \Theta^{(k(n))} \}, \{ p^{(k(n))} \}, \{ q^{(k(n))} \}$  such that  $\Theta^{(k(n))} \rightarrow u, p^{(k(n))} \rightarrow u_1$ , and  $q^{(k(n))} \rightarrow u_2$ , where  $u$  is a solution to the boundary value problem. Let us introduce the following definition.

**Definition 1.** A family of functions  $\{g^{(n)}(x,y)\}$  is equioscillating to zero on  $R$  if for any  $\epsilon > 0$  there is a  $\delta > 0$  and an  $n_0 > 0$  such that  $|g^{(n)}(\bar{x},\bar{y}) - g^{(n)}(x,y)| < \epsilon$  whenever  $n > n_0$ ,  $|\bar{x}-x| < \delta$ ,  $|\bar{y}-y| < \delta$ ,  $(\bar{x},\bar{y})$  in  $R$  and  $(x,y)$  in  $R$ .

We want to show that  $\{\Theta^{(n)}\}$ ,  $\{p^{(n)}\}$ ,  $\{q^{(n)}\}$  are uniformly bounded in absolute value and are equioscillating to zero on  $R$ .

**Theorem 4.1.**  $\{\Theta^{(n)}\}$ ,  $\{p^{(n)}\}$ ,  $\{q^{(n)}\}$  are uniformly bounded in absolute value on  $R$ .

**Proof:** We will first prove the theorem for  $\{\Theta^{(n)}\}$ ,  $\{\pi^{(n)}\}$ ,  $\{\rho^{(n)}\}$ .

From the definitions of  $\Theta$ ,  $\pi$ ,  $\rho$  (dropping the superscript  $n$ ) we see that there is a constant  $M_1 > 0$  such that

$$||\pi|| \leq M_1 + ||a|| ||\rho|| + ||b|| ||\Theta||,$$

$$||\rho|| \leq M_1 + ||c|| ||\pi|| + ||d|| ||\Theta||,$$

$$||\Theta|| \leq M_1 + A ||\pi|| + B ||\rho||.$$

Hence

$$||\pi|| \leq M_1 + ||a|| \{M_1 + ||c|| ||\pi|| + ||d|| ||\Theta||\} + ||b|| ||\Theta||,$$

and since  $||a|| ||c|| < 1$ ,

$$||\pi|| \leq \{M_1(1+||a||) + (||a|| ||d|| + ||b||) ||\Theta||\} \{1 - ||a|| ||c||\}^{-1}.$$

Similarly

$$||\rho|| \leq \{M_1(1+||c||) + (||b|| ||c|| + ||d||) ||\Theta||\} \{1 - ||a|| ||c||\}^{-1}.$$

Therefore,

$$||\Theta|| \leq M_1 \left\{ 1 + \frac{A(1+||a||) + B(1+||c||)}{1 - ||a|| ||c||} \right\} + \left\{ A(||a|| ||d|| + ||b||) + B(||b|| ||c|| + ||d||) \right\} \frac{||\Theta||}{1 - ||a|| ||c||}$$



and so by condition (h) of theorem (2.1) the  $||\theta||$ 's are uniformly bounded, and hence so are the  $||\pi||$ 's and the  $||\rho||$ 's.

But it is obvious that the uniform boundedness of the  $||\pi||$ 's implies the uniform boundedness of the  $||p||$ 's, and similarly for the  $||\rho||$ 's and  $||q||$ 's.

In order to show that the sequences are equioscillating to zero we need the following result.

Theorem 4.2. If  $0 \leq g(x) \leq M$  and  $0 \leq g(x) \leq \gamma + L \left| \int_a^x g(s) ds \right|$  for  $0 \leq a \leq A$ ,  $0 \leq x \leq A$ , then  $g(x) \leq \gamma \exp(LA)$ .

Proof: If  $a < x$  then

$$g(x) \leq \gamma + L \int_a^x \left\{ \gamma + L \int_a^{s_1} g(s_1) ds_1 \right\} ds \leq \gamma(1+LA) + L^2 \int_a^x ds \int_a^s g(s_1) ds_1 ,$$

$$\leq \dots \leq \gamma \sum_{n=0}^N \frac{(LA)^n}{n!} + M \frac{(LA)^{N+1}}{(N+1)!} .$$

Theorem 4.3. Each of the sequences  $\{\theta^{(n)}\}$ ,  $\{p^{(n)}\}$ ,  $\{q^{(n)}\}$  is equioscillating to zero on R.

Proof: The theorem is obviously true for  $\{\theta^{(n)}\}$ . We next consider  $\{\pi^{(n)}\}$  and  $\{\rho^{(n)}\}$ . Let us assume  $(x,y)$  in  $R_{j,k}$  and  $(\bar{x},\bar{y})$  in  $R_{j,K}$  and  $j,J,k,K$  all  $\geq 1$ . Again suppressing the superscript  $n$ , we have

$$|\pi(\bar{x},\bar{y}) - \pi(x,y)| \leq |\pi(\bar{x},\bar{y}) - \pi(\bar{x},y)| + |\pi(\bar{x},y) - \pi(x,y)| .$$

It is obvious that given  $\varepsilon > 0$ , there is a  $\delta > 0$  such that (for  $n$  large enough)

$|\pi(\bar{x},\bar{y}) - \pi(\bar{x},y)| < \varepsilon/2$  whenever  $(\bar{y}-y) < \delta$ . Hence we need only consider

the term  $|\pi(\bar{x},y) - \pi(x,y)|$ . Similarly it is only necessary to consider

$|\rho(x,\bar{y}) - \rho(x,y)|$ . But then by (3.1), (3.2), and applying the Lipschitz condition to  $f$ ,

$$|\pi(\bar{x}, y) - \pi(x, y)| \leq \gamma + ||a|| |\rho(x, \eta(x_j)) - \rho(x, \eta(x_j))|$$

$$+ L \left| \int_{\eta(x)}^y |\pi(\bar{x}, t) - \pi(x, t)| dt \right|,$$

$$|\rho(x, \bar{y}) - \rho(x, y)| \leq \gamma + ||c|| |\pi(\xi(y_k), y) - \pi(\xi(y_k), y)|$$

$$+ L \left| \int_{\xi(y)}^x |\rho(s, \bar{y}) - \rho(s, y)| ds \right|,$$

where  $\gamma \geq 0$  is equioscillating to zero.

Therefore by theorem 4.2,

$$|\pi(\bar{x}, y) - \pi(x, y)| \leq \left\{ \gamma + ||a|| |\rho(x, \eta(x_j)) - \rho(x, \eta(x_j))| \right\} \exp(LA)$$

$$|\rho(x, \bar{y}) - \rho(x, y)| \leq \left\{ \gamma + ||c|| |\pi(\xi(y_k), y) - \pi(\xi(y_k), y)| \right\} \exp(LB)$$

If we choose constants  $\beta, \omega, \psi$  such that

$$0 \leq \gamma \exp(LA) = \beta, \quad 0 \leq \gamma \exp(LB) = \beta, \quad ||a|| \exp(LA) = \omega, \quad ||c|| \exp(LB) = \psi,$$

then

$$|\pi(x_j, y_k) - \pi(x_j, y_k)| \leq \beta + \omega |\rho[x_j, \eta(x_j)] - \rho[x_j, \eta(x_j)]|,$$

$$|\rho(x_j, y_k) - \rho(x_j, y_k)| \leq \beta + \psi |\pi[\xi(y_k), y_k] - \pi[\xi(y_k), y_k]|.$$

Repeated application of these last inequalities gives

$$|\pi(x_j, y_k) - \pi(x_j, y_k)| \leq \beta + \omega \left\{ \beta + \psi |\pi[\xi(\eta(x_j)), \eta(x_j)] - \pi[\xi(\eta(x_j)), \eta(x_j)]| \right\}$$

$$= \beta(1+\omega) + \omega\psi |\pi[\xi(\eta(x_j)), \eta(x_j)] - \pi[\xi(\eta(x_j)), \eta(x_j)]| \leq \beta(1+\omega) \sum_{r=0}^m (\omega\psi)^r +$$

$$(\omega\psi)^{m+1} \sup \left\{ |\pi(\bar{x}, y) - \pi(x, y)|; (\bar{x}, y) \in R, (x, y) \in R \right\}.$$

Thus if  $\omega\psi < 1$  then  $\{\pi^{(n)}\}$  is equioscillating to zero. But

$\omega\psi = ||a|| ||c|| \exp[L(A+B)]$ , and this is  $< 1$  by hypotheses (i) of theorem 2.1. We dispose of  $\{p^{(n)}\}$  in a similar way.

It is obvious from the definition of  $p, q$  that since  $\{\pi^{(n)}\}, \{\rho^{(n)}\}$  are equioscillating to zero on  $R$ , then so are  $\{p^{(n)}\}, \{q^{(n)}\}$ .

### 5. Convergence to a solution.

Since each of the sequences  $\{\theta^{(n)}\}, \{p^{(n)}\}, \{q^{(n)}\}$  is equibounded and equioscillating to zero, we can apply Arzela's theorem. More specifically we can find a subsequence  $\{n(s)\}_{s=1}^{\infty}$  of  $\{n\}_{n=1}^{\infty}$  such that  $\theta^{(n(s))}$  converges uniformly to a continuous limit function  $\theta^*$  on  $R$ . Then we can find a subsequence  $\{n(s(t))\}_{t=1}^{\infty}$  of  $\{n(s)\}$  such that  $p^{(n(s(t)))}$  converges uniformly to a continuous limit function  $p^*$  on  $R$ . But then we can find a subsequence  $\{n(s(t(k)))\}_{k=1}^{\infty}$  of  $\{n(s(t))\}$  such that  $q^{(n(s(t(k))))}$  converges uniformly to a continuous limit function  $q^*$  on  $R$ . If we now choose the original subsequence  $\{n(s)\}_{s=1}^{\infty}$  to coincide with  $\{n(s(t(k)))\}_{k=1}^{\infty}$  then for  $(x, y)$  in  $R$

$$\lim_{s \rightarrow \infty} \theta^{(n(s))}(x, y) = \theta^*(x, y),$$

$$\lim_{s \rightarrow \infty} p^{(n(s))}(x, y) = p^*(x, y),$$

$$\lim_{s \rightarrow \infty} q^{(n(s))}(x, y) = q^*(x, y).$$

We want to show that  $\theta^*(x, y)$  is a solution of the boundary value problem, and that

$$\frac{\partial \theta^*}{\partial x} = p^*, \quad \frac{\partial \theta^*}{\partial y} = q^*.$$

To this end consider

$$(5.1) \quad \Theta^{(n(j))}(x,y) = \left\{ \int_0^x [\sigma(s) + a(s)q^*(s, \eta(s)) + b(s) \Theta^*(s, \eta(s))] ds \right. \\ \left. + \int_0^y [\tau(t) + c(t)p^*(\xi(t), t) + d(t) \Theta^*(\xi(t), t)] dt \right. \\ \left. + \left[ \int_0^y dt \int_{\xi(t)}^x ds - \int_0^x ds \int_0^{\eta(s)} dt \right] f[s, t, \Theta^*(s, t), p^*(s, t), q^*(s, t)] + u_0 \right\}.$$

Suppressing the superscript on  $\Theta^{(n(j))}$  and using the fact that

$$p[\lambda(x), \mu(y)] = \pi(x, y), \quad q[\lambda(x), \mu(y)] = \rho(x, y),$$

the above expression is just equal to

$$\int_0^x \left\{ \sigma(\lambda(s)) - \sigma(s) + a(\lambda(s))q[\lambda(s), \eta(\lambda(s))] - a(s)q^*(s, \eta(s)) \right. \\ \left. + b(\lambda(s)) \Theta[\lambda(s), \eta(\lambda(s))] - b(s)\Theta^*(s, \eta(s)) \right\} ds \\ + \int_0^y \left\{ \tau(\mu(t)) - \tau(t) + c(\mu(t))p[\xi(\mu(t)), \mu(t)] - c(t)p^*(\xi(t), t) \right. \\ \left. + d(\mu(t))\Theta[\xi(\mu(t)), \mu(t)] - d(t) \Theta^*(\xi(t), t) \right\} dt \\ + \left\{ \int_0^y dt \int_{\xi(t)}^x ds - \int_0^x ds \int_0^{\eta(s)} dt \right\} \left\{ f[\lambda(s), \mu(t), \Theta(\lambda(s), \mu(t)), \right. \\ \left. p(\lambda(s), \mu(t)), q(\lambda(s), \mu(t))] - f[s, t, \Theta^*(s, t), p^*(s, t), q^*(s, t)] \right\} + \gamma_1$$

$$\text{where } |\gamma_1| \leq M \left\{ B \sup_{0 \leq t \leq B} |\xi[\mu(t)] - \xi(t)| + A \sup_{0 \leq s \leq A} |\eta[\lambda(s)] - \eta(s)| \right\},$$

and this last term goes to zero as the  $\alpha_n$  (defined by (3.3)) goes to zero.

The absolute value of the integrand in the double integrals is not greater than

$$|f[\lambda, \mu, \Theta(\lambda, \mu), p(\lambda, \mu), q(\lambda, \mu)] - f[\lambda, \mu, \Theta^*(\lambda, \mu), p^*(\lambda, \mu), q^*(\lambda, \mu)]| \\ + |f[\lambda, \mu, \Theta^*(\lambda, \mu), p^*(\lambda, \mu), q^*(\lambda, \mu)] - f[s, t, \Theta^*(s, t), p^*(s, t), q^*(s, t)]| .$$

By the uniform convergence of  $\Theta, p, q$  to  $\Theta^*, p^*, q^*$  respectively, and by the continuity of  $f, \sigma, \tau, a, b, c, d$  it is clear that (5.1) goes to zero as  $n(j) \rightarrow \infty$ .

All that remains to do is to show that

$$\frac{\partial}{\partial x} \Theta^* = p^*, \quad \frac{\partial}{\partial y} \Theta^* = q^* .$$

To do this proceed with  $p$  and  $q$  in the same way as we did with  $\Theta$ , i.e., consider for example

$$p^{(n(j))}(x, y) - \left\{ \sigma(x) + a(x)q^*(x, \eta(x)) + b(x)\Theta^*(x, \eta(x)) \right. \\ \left. + \int_{\eta(x)}^y f[x, t, \Theta^*(x, t), p^*(x, t), q^*(x, t)] dt \right\} .$$

As in the case of (5.1) this expression goes to zero as  $n(j) \rightarrow \infty$ . Hence

$$\lim_{j \rightarrow \infty} p^{(n(j))} = p^* = \frac{\partial \Theta^*}{\partial x} .$$

Similarly we can show that  $q^* = \frac{\partial \Theta^*}{\partial y}$ . This completes the proof of theorem 2.1.

## 6. Other conditions on $f$ .

It was shown by Glick [3] that in the case of the  $n$ -dimensional characteristic boundary value problem the condition that  $|f|$  be bounded can be replaced by the condition that  $f$  satisfy a Lipschitz condition in  $u$  as well

as in the partial derivatives of  $u$ . The following theorem shows that we can get a similar result for problem (1.3).

**Theorem 6.1.** We make the same assumptions in theorem 2.1, except that conditions (b) and (c) are replaced by the following condition:

$f$  satisfies a uniform Lipschitz condition in  $(u, p, q)$  i.e., there is a constant  $L > 0$  such that  $|f(x, y, \bar{u}, \bar{p}, \bar{q}) - f(x, y, u, p, q)| \leq L(|\bar{u} - u| + |\bar{p} - p| + |\bar{q} - q|)$  whenever the arguments of  $f$  are in  $C$ .

Then for  $A, B$  sufficiently small, we have the same conclusion as in theorem 2.1.

**Proof:** Let  $\bar{f}(x, y, u, p, q) = f(x, y, 0, 0, 0)$  and let  $\bar{\Theta}, \bar{\pi}, \bar{\rho}$  denote  $\Theta, \pi, \rho$  corresponding to the function  $\bar{f}$ . Then since  $|\bar{f}|$  is bounded on  $R$  and hence on  $C$ , the sequences  $\{\bar{\Theta}^{(n)}\}, \{\bar{\pi}^{(n)}\}, \{\bar{\rho}^{(n)}\}$  are uniformly bounded on  $R$ . Hence there is a constant  $M > 0$  such that

$$\begin{aligned} ||\pi|| &\leq ||\bar{\pi}|| + ||\pi - \bar{\pi}|| \leq ||\bar{\pi}|| + ||a|| ||\rho - \bar{\rho}|| + ||b|| ||\Theta - \bar{\Theta}|| \\ &+ L \left| \int_{\eta}^y \{ ||\Theta|| + ||\pi|| + ||\rho|| \} dt \right| \\ &\leq M + ||a|| ||\rho|| + ||b|| ||\Theta|| + LB(||\Theta|| + ||\pi|| + ||\rho||). \end{aligned}$$

Similarly

$$||\rho|| \leq M + ||c|| ||\pi|| + ||d|| ||\Theta|| + LA(||\Theta|| + ||\pi|| + ||\rho||),$$

and

$$\begin{aligned} ||\Theta|| &\leq M + A(||a|| ||\rho|| + ||b|| ||\Theta||) + B(||c|| ||\pi|| + ||d|| ||\Theta||) \\ &+ LAB(||\Theta|| + ||\pi|| + ||\rho||). \end{aligned}$$

It is obvious that for A,B sufficiently small we can proceed as in section 4 to prove the uniform boundedness of  $\Theta^{(n)}$ ,  $\pi^{(n)}$ ,  $\rho^{(n)}$ . The rest of the proof goes through as before.

Finally we note that the boundedness of  $|f|$  or the Lipschitz continuity of  $f$  with respect to  $u$  can be replaced by the condition

$$|f| < K(|u| + |p| + |q|)$$

for some constant  $K > 0$ . To see this it is only necessary to point out that in this case we have

$$\begin{aligned} ||\pi|| &\leq ||\sigma|| + ||a|| ||\rho|| + ||b|| ||\Theta|| + KB(||\Theta|| + ||\pi|| + ||\rho||), \\ ||\rho|| &\leq ||\pi|| + ||c|| ||\pi|| + ||d|| ||\Theta|| + KA(||\Theta|| + ||\pi|| + ||\rho||), \\ ||\Theta|| &\leq A(||\sigma|| + ||a|| ||\rho|| + ||b|| ||\Theta||) + B(||\pi|| + ||c|| ||\pi|| + ||d|| ||\Theta||) \\ &\quad + KAB(||\Theta|| + ||\pi|| + ||\rho||). \end{aligned}$$

and again, for A,B sufficiently small it follows that  $\{\Theta^{(n)}\}$ ,  $\{\pi^{(n)}\}$ ,  $\{\rho^{(n)}\}$  are uniformly bounded.

This kind of condition on  $f$  has been used by Z. Szmydt [5] who treats a similar problem by a fixed point technique.

REFERENCES

- [1] Aziz, A. K., and Diaz, J. B.: On a mixed boundary value problem for linear hyperbolic partial differential equations in two independent variables. Archive for Rational Mechanics and Analysis. Vol. 10, No. 1, 1-28(1962).
- [2] Conlan, J.: The Cauchy Problem and the mixed boundary value problem for a nonlinear hyperbolic partial differential equation in two independent variables. Archive for Rational Mechanics and Analysis. Vol. 3, No. 4, 355-380(1959).
- [3] Diaz, J. B.: On an analogue of the Euler-Cauchy polygon method for the numerical solution of  $u_{xy} = F(x, y, u, u_x, u_y)$ . Archive for Rational Mechanics and Analysis. Vol. 1, No. 4, 357-390(1958).
- [4] Glick, I. I.: On an analogue of the Euler-Cauchy polygon method for the partial differential equation  $u_{x_1 \dots x_n} = f$ . NOLTR 61-26, U. S. Naval Ordnance Laboratory, April 26, 1961.
- [5] Szmydt, Z.: Sur l'existence de solutions de certains problemes aux limites a un systems d'equations differentielles hyperboliques. Bulletin de l'Academie Polonaise des Sciences, serie des math., astr., et phys. Vol. VI, No. 1, 1958.



NOLTR 62-177

EXTERNAL DISTRIBUTION LIST (M1)

|  | Copies |  | Copies |
|--|--------|--|--------|
| Chief, Defense Atomic Support Agency<br>Washington 25, D. C.<br>Attn: Document Library Branch  | 2      | Director of Intelligence<br>Headquarters, USAF<br>Washington 25, D. C.<br>Attn: AOIN-3B  |        |
| Commanding General<br>Aberdeen Proving Ground, Md.<br>Attn: Tech. Information Branch   | 2      | Langley Research Center<br>Hampton, Virginia<br>Attn: Theoretical Aero-<br>dynamics Division   | 1      |
| Attn: Director, Ballistic<br>Research Laboratories   | 2      | Attn: Librarian  | 2      |
| National Aeronautics and Space Agency<br>Geo. C. Marshall Space Flight Center<br>Huntsville, Alabama<br>Attn: M-SandS-PT<br>(Mr. H. A. Connell)                      | 3      | Attn: Adolf Busemann   | 1      |
| ASTIA<br>Arlington Hall<br>Arlington, Va.  | 10     | Attn: John Stack   | 2      |
| Chief, Bureau of Naval Weapons<br>Department of the Navy<br>Washington 25, D. C.<br>Attn: DLI-3  | 4      | National Aeronautics and Space Agency<br>1512 H Street, N. W.<br>Washington 25, D. C.<br>Attn: B. A. Mulcahy, Chief<br>Div. of Research<br>Information |        |
| Attn: RR-14  | 1      | Ames Research Center<br>Moffett Field, California<br>Attn: Librarian   |        |
| Attn: RMMS   | 1      | NASA Flight Research Center<br>Edwards, California<br>Attn: Mr. W. C. Williams   |        |
| Attn: RRRE   | 1      | Lewis Research Center<br>21000 Brookpark Road<br>Cleveland, 11, Ohio<br>Attn: Librarian  |        |
| Attn: RRMA   | 1      | Attn: Chief, Supersonic<br>Propulsion Div.   |        |
| Attn: RMGA   | 1      | Commander,<br>Pacific Missile Range<br>Point Mugu, California<br>Attn: Technical Library   | 2      |
| Commanding Officer and Director<br>David Taylor Model Basin<br>Aerodynamics Laboratory<br>Washington 7, D. C.<br>Attn: Library                                       |        | National Bureau of Standards<br>Washington 25, D. C.<br>Attn: Applied Math. Div.   |        |
| Commanding Officer and Director<br>David Taylor Model Basin<br>Applied Mathematics Laboratory<br>Washington 7, D. C.<br>Attn: Dr. H. Polachek<br>Attn: Dr. D. Shanks |        | Office of Technical Services<br>Department of Commerce<br>Washington 25, D. C.   | 100    |
| Commanding Officer<br>Diamond Ordnance Fuze Laboratory<br>Washington 25, D. C.<br>Attn: Library, Bldg. 92, Rm.211  |        |  |        |

NOLTR 62-177

EXTERNAL DISTRIBUTION LIST (M1)

|  | Copies |  | Copies |
|--|--------|--|--------|
| Commander<br>U. S. Naval Ordnance Test Station<br>China Lake, California<br>Attn: Technical Library<br>Attn: Code 406<br>Attn: Code 507<br>Attn: Code 5019 |        | The Avco Manufacturing Corp.,<br>Research Laboratories<br>2385 Revere Beach Pkwy.<br>Everett 49, Massachusetts   |        |
| Commander<br>U. S. Naval Weapons Laboratory<br>Dahlgren, Virginia<br>Attn: Technical Library   | 2      | CONVAIR Corp.,<br>A Division of General Dynamics<br>Corp.,<br>Daingerfield, Texas.   |        |
| Commanding Officer<br>ONR Branch Office<br>Box 39, Navy 100<br>Fleet Post Office<br>New York, New York   |        | Cornell Aeronautical Lab. Inc.,<br>P. O. Box 235, 4455 Genessee St.<br>Buffalo 21, New York<br>Attn: Librarian   |        |
| Office of Naval Research<br>T-3 Building, Room 2715<br>Washington 25, D. C.<br>Attn: Head, Mechanics Branch<br>Attn: Dr. J. Weyl                           |        | Douglas Aircraft Company, Inc.,<br>Santa Monica Division<br>3000 Ocean Park Blvd.,<br>Santa Monica, California<br>Attn: Chief Engineer                   |        |
| Naval Research Laboratory<br>Washington 25, D. C.<br>Attn: Dr. H.M.Trent, Code 6230<br>Attn: Code 2027   |        | Navy Ordnance Division<br>Eastman Kodak Company<br>50 West Main Street<br>Rochester 14, New York<br>Attn: Mr. W. B. Forman                               | 2      |
| Commander<br>Aeronautical Systems Division<br>Air Force Systems Command<br>Wright-Patterson AF Base, Ohio<br>Attn: WCOSI-3                                 |        | Missile and Ordnance Systems<br>Dept.,<br>General Electric Co.,<br>3198 Chestnut Street<br>Philadelphia 4, Pennsylvania<br>Attn: L. Chasen, Mgr. Library | 2      |
| Oak Ridge National Laboratory<br>Oak Ridge, Tennessee<br>Att: Dr. A. S. Householder  |        | Mathematics Research Center<br>U. S. Army<br>University of Wisconsin<br>Madison, Wisconsin<br>Attn: Prof. R.E.Langer<br>Attn: Dr. Donald Greenspan       |        |
| AER, Inc.,<br>871 E. Washington Street<br>Pasadena, California<br>Attn: Dr.A.J.A.Morgan  |        |  |        |
| Allied Research Associates, Inc.,<br>43 Leon Street<br>Boston 15, Massachusetts<br>Attn: Miss Joyce Larkin   |        |  |        |

EXTERNAL DISTRIBUTION LIST (ML)

| <u>No. of<br/>Copies</u> |   | <u>No. of<br/>Copies</u> |  |
|--------------------------|---|--------------------------|--|
|                          | Guggenheim Aeronautical Laboratory<br>California Institute of<br>Technology<br>Pasadena 4, California |                          | Case Institute of Technology<br>Cleveland 6, Ohio  |
| 1                        | Attn: Aeronautics Library   | 1                        | Attn: G. Kuerti  |
|                          | Hughes Aircraft Co.<br>Florence Avenue at Teale Street<br>Culver City, California                     |                          | Cornell University<br>Graduate School of Aero. Eng.<br>Ithaca, New York  |
| 1                        | Attn: Mr. Dana H. Johnson<br>R & D Tech Library   | 1                        | Attn: Prof. W. R. Sears  |
|                          | McDonnell Aircraft Corp.<br>P. O. Box 516<br>St. Louis 3, Missouri                                    |                          | Applied Physics Laboratory<br>Johns Hopkins University<br>8621 Georgia Avenue<br>Silver Spring, Maryland   |
| 1                        | Attn: Engineering Library,<br>Dept. 644   | 2                        | Attn: Technical Reports Group  |
|                          | United Aircraft Corporation<br>400 Main Street<br>East Hartford 8, Connecticut                        |                          | The Johns Hopkins University<br>Charles and 34th Streets<br>Baltimore 18, Maryland   |
| 1                        | Attn: Chief Librarian   | 1                        | Attn: Dr. Francis H. Clauser   |
|                          | Superintendent<br>U. S. Naval Postgraduate School<br>Monterey, California                             |                          | Mathematics Department<br>Harvard University<br>Cambridge 39, Massachusetts  |
| 1                        | Attn: Library, Technical<br>Reports Section   | 1                        | Attn: Prof. G. Birkhoff  |
|                          | University of California<br>Berkeley 4, California  | 2                        | Director<br>Institute for Fluid Dynamics and<br>Applied Mathematics<br>University of Maryland<br>College Park, Maryland  |
| 1                        | Attn: Dr. S. A. Schaaf  |                          | Massachusetts Institute of Tech.<br>Cambridge 39, Massachusetts  |
|                          | University of California<br>Los Angeles 24, California  | 1                        | Attn: Prof. Joseph Kaye,<br>Room 1-212   |
| 1                        | Attn: Dr. G. E. Forsythe  |                          | University of Michigan<br>Randall Laboratory<br>Ann Arbor, Michigan  |
|                          | California Institute of Technology<br>Pasadena 4, California  | 1                        | Attn: Prof. Otto Laporte   |
| 1                        | Attn: Aeronautics Department  |                          | University of Michigan<br>Supervisor<br>Technical Documents Services<br>Willow Run Laboratories<br>University of Michigan<br>Willow Run Airport<br>Ypsilanti, Michigan |
|                          | Jet Propulsion Laboratory<br>California Institute of Technology<br>Pasadena 4, California             |                          |  |
| 1                        | Attn: Library   |                          |  |

NOLTR 62-177

EXTERNAL DISTRIBUTION LIST (M1)

No. of  
Copies

|   |  |
|---|--|
| 1 | University of Minnesota<br>Minneapolis 14, Minnesota<br>Attn: Mechanical Eng. Dept.  |
| 1 | The Ohio State University<br>Research Foundation<br>Columbus 10, Ohio<br>Attn: Security Officer,<br>Carlos Brewer                      |
| 1 | Attn: Dr. Paul M. Pepper   |
| 1 | Applied Math. & Statistics Lab.<br>Stanford University<br>Stanford, California   |
| 1 | Defense Research Laboratory<br>The University of Texas<br>P. O. Box 8092<br>Austin 12, Texas<br>Attn: H. D. Krick, Asst. Director      |
| 1 | Department of Civil Engineering<br>Colorado State University<br>Fort Collins, Colorado<br>Attn: Professor J. E. Cermak                 |
| 1 | Electrical Engineering Department<br>Johns Hopkins University<br>Baltimore 18, Maryland<br>Attn: Dr. N. H. Choksy                      |
|   | General Electric Company<br>Missile and Space Division<br>3198 Chestnut Street<br>Philadelphia, Pennsylvania<br>Attn: Mr. Jerome Persh |

CATALOGING INFORMATION FOR LIBRARY USE

BIBLIOGRAPHIC INFORMATION

|               | DESCRIPTORS          | CODES  | SECURITY CLASSIFICATION<br>AND CODE COUNT  | DESCRIPTORS       | CODES |
|---------------|----------------------|--------|--|-------------------|-------|
| SOURCE        | NOL technical report | NOLTR  |  | Unclassified - 19 | U019  |
| REPORT NUMBER | 62-177               | 620177 | CIRCULATION LIMITATION                     |                   |       |
| REPORT DATE   | 17 October 1962      | 1062   | CIRCULATION LIMITATION<br>OR BIBLIOGRAPHIC |                   |       |
|               |                      |        | BIBLIOGRAPHIC<br>(SUPPL., VOL., ETC.)      |                   |       |

SUBJECT ANALYSIS OF REPORT

|              | DESCRIPTORS | CODES | DESCRIPTORS   | CODES |
|--------------|-------------|-------|---------------|-------|
| Partial      |             | PARI  | Sequences     | SEQU  |
| Differential |             | DIFE  | Approximating | APPR  |
| Equation     |             | EQUA  | Functions     | FUNC  |
| Theorem      |             | THEY  | Convergence   | CONVG |
| Generalized  |             | GENA  | Solution      | SOLG  |
| Boundary     |             | BOUD  | Euler         | EULE  |
| Value        |             | VALU  | Cauchy        | CAUC  |
| Problem      |             | PROB  |               |       |
| Finite       |             | FNIT  |               |       |
| Linear       |             | LINN  |               |       |
| Combinations |             | CMBI  |               |       |
| Mathematical |             | MATH  |               |       |

|  |  |
|--|--|
| <p>Naval Ordnance Laboratory, White Oak, Md.<br/>(NOL technical report 62-177)<br/>A GENERALIZED BOUNDARY VALUE PROBLEM FOR<br/><math>u_{xy}=f(x,y,u,u_x,u_y)</math>, by James Conlan. 17 Oct.<br/>1962. 10p. tables. NOL task FR-30.<br/>UNCLASSIFIED</p> <p>This paper proves an existence theorem for the partial differential equation <math>u_{xy}=f(x,y,u,u_x,u_y)</math> with certain linear combinations of <math>u,u_x</math>, and <math>u_y</math> given on two non characteristic curves.</p> | <ol style="list-style-type: none"> <li>1. Differential equations, Partial</li> <li>2. Euler-Cauchy method</li> <li>3. Hyperbolic systems</li> </ol> <p>I. Title<br/>II. Conlan, James<br/>III. Project</p> |
| <p>Naval Ordnance Laboratory, White Oak, Md.<br/>(NOL technical report 62-177)<br/>A GENERALIZED BOUNDARY VALUE PROBLEM FOR<br/><math>u_{xy}=f(x,y,u,u_x,u_y)</math>, by James Conlan. 17 Oct.<br/>1962. 10p. tables. NOL task FR-30.<br/>UNCLASSIFIED</p> <p>This paper proves an existence theorem for the partial differential equation <math>u_{xy}=f(x,y,u,u_x,u_y)</math> with certain linear combinations of <math>u,u_x</math>, and <math>u_y</math> given on two non characteristic curves.</p> | <ol style="list-style-type: none"> <li>1. Differential equations, Partial</li> <li>2. Euler-Cauchy method</li> <li>3. Hyperbolic systems</li> </ol> <p>I. Title<br/>II. Conlan, James<br/>III. Project</p> |
| <p>Naval Ordnance Laboratory, White Oak, Md.<br/>(NOL technical report 62-177)<br/>A GENERALIZED BOUNDARY VALUE PROBLEM FOR<br/><math>u_{xy}=f(x,y,u,u_x,u_y)</math>, by James Conlan. 17 Oct.<br/>1962. 10p. tables. NOL task FR-30.<br/>UNCLASSIFIED</p> <p>This paper proves an existence theorem for the partial differential equation <math>u_{xy}=f(x,y,u,u_x,u_y)</math> with certain linear combinations of <math>u,u_x</math>, and <math>u_y</math> given on two non characteristic curves.</p> | <ol style="list-style-type: none"> <li>1. Differential equations, Partial</li> <li>2. Euler-Cauchy method</li> <li>3. Hyperbolic systems</li> </ol> <p>I. Title<br/>II. Conlan, James<br/>III. Project</p> |